International Journal of Theoretical Physics, Vol. 12, No. 1 (1975), pp. 1-9

Derivation of a Mathematical Equation for the EEG and the General Solution Within the Brain and in Space

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Received: 8 August 1973

Abstract

A mathematical model for the EEG is presented here based on physical and anatomical characteristics of the brain. The important finding is that the system is characterised by a diffusion equation whose general solution represents a Fourier series expansion for the EEG at any point of the zone under consideration.

1. *Introduction*

Evidence from anatomical studies suggests that the nerve cells in the cerebral cortex are grouped in layers and that within these layers they are more or less uniformly distributed. Under this hypothesis we shall construct a simplified model of the EEG. Let us assume that the brain tissue is an isotropic, diffusive medium in which are uniformly embedded throughout a great many unit voltage generators, viz., neurons and glia. These generators can generate signals which are identical in wave form, but the time of generation of a signal by any particular unit source is an independent event. This volume conductor is electrically continuous with other tissues—the skull, connective tissue, etc.—which do not generate electrical activity of the same type. We also assume that the passive elements present in the medium are electrically only capacitive or resistive. Given this simplified model, we are interested in finding the form of brain potentials produced by these generators which we shall observe at any particular point of this medium.

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2. *Derivation of the Equation for the Electrical Potential*

The properties of the medium enable us to use the following equations for the isotropic media, viz.:

$$\nabla . \mathbf{D} = \rho \tag{2.1}$$

$$\mathbf{J} = \sigma \mathbf{E} \tag{2.2}$$

$$\mathbf{D} = \epsilon \mathbf{E} \tag{2.3}$$

Where the vectors **D**, **E**, and **J** are the electrical displacement, electrical field and the current density, respectively. ρ, the free charge density; σ, the conductivity, and ϵ, the permittivity of the medium are constants and independent of the field intensities and spacial coordinates.

Let C be the capacitivity (Farad/cm^3) of the medium. Since Charge = Potential difference x Capacitance we can write

$$\nabla \left(\frac{\rho}{\epsilon}\right) = -\frac{C\mathbf{E}}{\epsilon} \tag{2.4}$$

Assuming the law of conservation of charges we have

$$\nabla . \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{2.5}$$

Using (2.1), (2.2) and (2.3) we have

$$\frac{\rho}{\epsilon} = \nabla . \mathbf{E} \tag{2.6}$$

Consequently (2.5) can be written as

$$\nabla . \mathbf{E} = -\frac{\epsilon}{\sigma}\frac{\partial}{\partial t}(\nabla . \mathbf{E}) \tag{2.7}$$

Equation (2.4) leads to

$$\nabla(\nabla . \mathbf{E}) = -\frac{C\mathbf{E}}{\epsilon} \tag{2.8}$$

Operating (2.6) on both sides by ∇ and substituting the result from (2.8) we get

$$\nabla(\nabla . \mathbf{E}) = \frac{C}{\sigma}\frac{\partial \mathbf{E}}{\partial t} \tag{2.9}$$

If we neglect the inductive effects, the field equation (2.9) may be considered irrotational and hence we could substitute

$$\mathbf{E} = -\nabla \phi \tag{2.10}$$

into (2.9) and choosing the initial conditions so that the constant term is zero, we get

$$\nabla^2 \phi = \frac{C}{\sigma} \frac{\partial \phi}{\partial t} \tag{2.11}$$

where we have assumed that C, the capacitivity is a local constant and that conditions are met so that the operators $\partial/\partial t$ and ∇ commute.

Equation (2.11) is the well-known form of the diffusion equation. The solution of equation (2.11) under simple boundary conditions shows that the medium applies strong attenuation on any signal as a function of frequency and distances from the source. Using the above equation in cylindrical coordinates, we can find the potential at any point within the medium (Bateman, 1959). The diffusion equation in cylindrical coordinates becomes:

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \psi^2} + \frac{\partial^2 \phi}{\partial Z^2} = \frac{C}{\sigma} \frac{\partial \phi}{\partial t} \tag{2.12}$$

For a two-dimensional solution, we may omit $\partial^2\phi/\partial Z^2$ and if $C/\sigma = 1/\delta$ (where δ is the diffusion parameter which depends weakly on time), we get:

$$\frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \psi^2} = \frac{1}{\delta} \frac{\partial \phi}{\partial t} \tag{2.13}$$

3. *Solution of the Diffusion Equation*

Let

$$\Phi(\omega, r, t, \psi) = \mathrm{e}^{-jwt} \Phi^*(r, \omega, m)\, \mathrm{e}^{jm\psi} \tag{3.1}$$

be the solution of (2.13).

Substituting (3.1) into (2.13), we get:

$$\frac{d^2\phi^*}{dr^2} + \frac{1}{r} \frac{d\phi^*}{dr} + \left(\frac{j\omega}{\delta} - \frac{m^2}{r^2}\right)\phi^* = 0 \tag{3.2}$$

The solution of (3.2) can be obtained easily and is of the form

$$\Phi^* = \alpha_m(\omega, \delta) J_m\left[\sqrt{\left(\frac{j\omega}{\delta}\right)}\, r\right] \tag{3.3}$$

Thus, (3.1) reduces to

$$\Phi^* = \alpha_m(\omega, \delta)\, \mathrm{e}^{j(m\psi - \omega t)} J_m\left[\sqrt{\left(\frac{j\omega}{\delta}\right)}\, r\right] \tag{3.4}$$

where $J_m(\cdot)$ is the Bessel function of order m.

Hence, the general solution is:

$$\Phi = \sum_{m=0}^{\infty} \sum_{\omega} \alpha_m(\omega, \delta)\, e^{j(m\psi - \omega t)} J_m\left[\sqrt{\left(\frac{j\omega}{\delta}\right)} r\right] \tag{3.5}$$

If we substitute the frequency f for the angular frequency ω, we get

$$\Phi(f, r, t, \psi) = \sum_{m=0}^{\infty} \sum_{f=-\infty}^{\infty} \alpha_m(f, \delta)\, e^{j(m\psi - 2\pi f t)} J_m\left[\sqrt{\left(\frac{2\pi f j}{\delta}\right)} r\right] \tag{3.6}$$

Here $J_m(\cdot)$ is the Bessel function of order m which is given by the following expression

$$J_m\left[\sqrt{\left(\frac{2\pi f j}{\delta}\right)} r\right] = \frac{j^{-m}}{\pi} \int_0^{\pi} e^{j^{3/2}} \sqrt{(2\pi f/\delta)} r \cos\theta \cos(m\theta)\, d\theta \tag{3.7}$$

Using (3.7), equation (3.6) takes the form

$$\Phi = \sum_{m=0}^{\infty} \sum_{f=-\infty}^{\infty} \frac{\alpha_m(f, \delta)}{\pi} j^{-m} \int_0^{\pi} \exp[-\sqrt{(\pi f/\delta)} r \cos\theta + j\sqrt{(\pi f/\delta)} r \cos\theta$$

$$+ m\psi - 2\pi f t] \cos(m\theta)\, d\theta \tag{3.8}$$

Separating the even and odd terms of m, we get

$$\Phi = \sum_{n=0}^{\infty} \sum_{f=-\infty}^{\infty} \frac{\alpha_n(f, \delta)}{\pi} (-1)^n \int_0^{\pi} \exp[-\sqrt{(\pi f/\delta)} r \cos\theta] \cos(2n\theta)$$

$$\times \{\cos[\sqrt{(\pi f/\delta)} r \cos\theta + 2\psi n - 2\pi f t]$$

$$-j \sin[\sqrt{(\pi f/\delta)} r \cos\theta + 2n\psi - 2\pi f t]\}\, d\theta$$

$$+ \sum_{n=0}^{\infty} \sum_{f=-}^{\infty} \frac{\beta_n(f, \delta)}{\pi} (-1)^n j \int_0^{\pi} \exp[-\sqrt{(\pi f/\delta)} r \cos\theta] \cos\overline{(2n-1\theta)}$$

$$\times \{\cos[\sqrt{(\pi f/\delta)} r \cos\theta + (2n+1)\psi - 2\pi f t] - j \sin[\sqrt{(\pi f/\delta)} r \cos\theta$$

$$+ (2n+1)\psi - 2\pi f t]\}\, d\theta \tag{3.9}$$

Finally, by taking the real part of (3.9), we get the solution for (2.13) thus:

$$\operatorname{Re}\Phi(f,r,t,\psi)=\sum_{n=0}^{\infty}\sum_{f=-\infty}^{\infty}\frac{(-1)^n}{\pi}\int_0^{\pi}\exp\left[-\sqrt{(\pi f/\delta)}\,r\cos\theta\right]$$

$$\times\{\alpha_n(f,\delta)\cos(2n\theta)\cos\left[\sqrt{(\pi f/\delta)}\,r\cos\theta+2n\psi-2\pi ft\right]$$

$$-\beta_n(f,\delta)\cos(\overline{2n-1}\theta)\sin\left[\sqrt{(\pi f/\delta)}\,r\cos\theta+(2n+1)\psi$$

$$-2\pi ft]\}\,d\theta \qquad (3.10)$$

where $\alpha_n(f, \delta)$ and $\beta_n(f, \delta)$ are arbitrary parameters.

4. *Discussion*

The last equation gives the general solution for the potential at any point in the brain. It is easily seen that the solution represents a Fourier expansion for the EEG at any point under the hypothesis that EEG is the summand of all potentials from all the generators embedded in the brain tissue. The coefficients $\alpha_n(f, \delta)$ and $\beta_n(f, \delta)$ can be obtained by matching the expansion with the coefficients of an observed EEG. The number of observations required

TABLE 1. $\alpha_0 = -100.\ (\times 10^3)$

f	a ($\times 10^{-3}$)	b ($\times 10^{-3}$)
1	−27·7	0·0807
2	−5·53	−0·153
3	−2·52	0·296
4	−2·05	0·0431
5	−0·208	0·0750
6	−0·456	0·0340
7	−0·180	0·0617
8	−0·292	0·0265
9	0·0299	0·0669
10	0·129	−0·0359
11	−0·118	0·0197
12	−0·140	−0·0197
13	−0·0731	0·0127
14	−0·0281	−0·0160
15	−0·103	−0·0220
16	−0·0562	0·0374
17	−0·00121	0·0223
18	0·0236	0·00684
19	0·0565	−0·0152
20	0·0583	−0·0391

for the solution will depend on the rate of convergence of the expansion of the series over n.

Our preliminary analysis consisted of a Fourier expansion of a normal human EEG record. The EEG data were digitised by the computer program (Walter, 1972). The coefficients a_f and b_f calculated in accordance with the simplified model (equation (A12)), from the EEG record over a period of 500 msecs are given in Table 1.

In Table 2 are shown the values of a_f and b_f calculated from another independent EEG record for the same individual under similar activity spanning again a period of 500 m sec.

A comparison of the corresponding coefficients in Tables 1 and 2 shows an overall similarity and stability of the coefficients for the same individual under essentially similar kinds of mental activity. Thus the set of (a_f, b_f) constitute a 'coordinate system' characteristic of a particular individual while under a particular kind of activity.

By virtue of (3.3) the coefficients are related to the value of the diffusion parameter 'δ' which in turn depends on the physiological parameters 'C' and σ. Consequently the sources of variation among the coefficients (a_f, b_f) between individuals and between various types of cerebral activity are capable of being directly related to the changes in 'δ', which we have assumed to be

TABLE 2. $\alpha_0 = -160 . (\times 10^3)$

f	a $(\times 10^{-3})$	b $(\times 10^{-3})$
1	−29·8	0·124
2	−8·11	−3·69
3	−2·97	0·354
4	−2·72	−0·128
5	−1·37	0·101
6	−1·25	0·228
7	−0·237	0·220
8	−0·416	0·0224
9	0·105	0·0785
10	0·116	−0·0770
11	−0·0922	0·0141
12	−0·134	−0·0122
13	0·00181	0·00985
14	0·0426	−0·0490
15	−0·116	−0·0703
16	−0·0852	0·0
17	−0·0260	−0.022
18	−0·0181	−0·0463
19	−0·0202	−0·0479
20	0·0314	−0·0358

weakly time dependent and of large magnitude (see Appendix) for these calculations. For a more rigorous description it would be necessary to take into account the contributions from the second-order terms obtained by retaining the first few orders of the Bessel function.

The theoretical model considered in this paper could be generalised to derive a more precise mathematical relationship by using Maxwell's equations for anisotropic media and also by considering 'δ' to be strongly time dependen The complexity of the results however do not appear to have an easy explanation in terms of simple physiological variables at the present stage. These relationships are currently being investigated by us along with the problem of the determination of a 'coordinate system' for pathological EEG records and our findings would be reported in subsequent publications.

Acknowledgement

The authors are grateful to Prof. R. Elul for making available his experimental set of EEG data. Dr. S. Raman is supported by the National Health and Welfare of Canada, Project 606-7-596.

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Appendix

The equation for electromagnetic wave propagation in free space is

$$\nabla^2 \phi - \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{A.1}$$

where $v = c/\sqrt{(\mu\epsilon)}$ and c is the velocity of light and ϵ, μ are the susceptibility and permeability for free space.

In cylindrical polar coordinates, equation (A.1) transforms to

$$\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \psi^2} + \frac{\partial^2 \phi}{\partial Z^2} = \frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2} \tag{A.2}$$

For a planar solution we can neglect $\partial^2\phi/\partial Z^2$.

Let us assume the solution as

$$\Phi = e^{-j\omega t}\Phi^*(r, \omega, m, v)\, e^{jm\psi} \tag{A.3}$$

Hence, Φ^* must satisfy the equation

$$\frac{d^2\phi^*}{dr^2} + \frac{1}{r}\frac{d\phi^*}{dr} - \frac{m^2}{r^2}\Phi^* + \frac{\omega^2}{v^2}\Phi^* = 0 \tag{A.4}$$

whose solution is

$$\Phi^*(r, \omega, m, v) = \alpha_m^*(\omega, v, r) J_m\left(\frac{\omega}{v} r\right) \tag{A.5}$$

Hence the general solution for equation (A.2) can be written as the real part of:

$$\Phi = \sum_{m=0}^{\infty} \sum_{\omega} \alpha_m^*(\omega, v, r)\, e^{j(m\psi - \omega t)} J_m\left(\frac{\omega}{v} r\right) \qquad r \geqslant R \tag{A.6}$$

where R is the bounding radius in the observed direction.

Substituting frequency 'f' for the angular frequency 'ω', the solution becomes:

$$\Phi = \sum_{m=0}^{\infty} \sum_{f=-\infty}^{\infty} \alpha_m^*(f, v, r)\, e^{j(m\psi - 2\pi f t)} j_m\left(\frac{2\pi f r}{v}\right) \qquad r \geqslant R \tag{A.7}$$

Equating the real parts of equations (A.7), (3.6)

$$\text{Re}\left\{\alpha_m e^{j(m\psi - 2\pi f t)} J_m\left[\frac{\sqrt{(2\pi f j)}}{\delta} R\right]\right\} = \text{Re}\left\{\alpha_m^*\, e^{j(m\psi - 2\pi f t)} j_m\left(\frac{2\pi f R}{v}\right)\right\} \tag{A.8}$$

we obtain the value for α_m^*.

The real part of equation (A.7) is

$$\text{Re}\,\Phi = \sum_{m=0}^{\infty} \sum_{f=-\infty}^{\infty} \alpha_m^* \cos(m\psi - 2\pi f t) J_m\left(\frac{2\pi f r}{v}\right) \tag{A.9}$$

$2\pi f r$ is the total distance around the circle of radius r in unit time.

Since the frequencies involved are very low, $(2\pi f r/v)$ is a small quantity and can be neglected, but

$$J_m\left(\frac{2\pi f r}{v}\right) = \left(\frac{\pi f r}{v}\right)^m \sum_{k=0}^{\infty} \left(\frac{-\pi^2 f^2 r^2}{v^2}\right)^k \Big/ k!(m+k)! \tag{A.10}$$

(Abramowitz & Stegun, 1964).

For

$$m = 0, J_0(\cdot) = 1 - 0\left(\frac{2\pi fr}{v}\right)^2$$

$$m \geqslant 1, J_m(\cdot) = 0\left(\frac{2\pi fr}{v}\right)$$

Hence:

$$\Phi = \sum_{f=-\infty}^{\infty} \alpha_0^* \, \mathrm{e}^{-j2\pi ft}$$

$$= \alpha_0' - \sum_{f=1}^{\infty} (\alpha_f' \cos 2\pi ft + \beta_f' \sin 2\pi ft) \qquad \text{(A.11)}$$

Since $1/\delta$ is very small $[0(10^{-6})]$ (we take values for the σ of the order 5.0×10^{-3} mho/cm (Ranck, 1963)), using expansion (A.10) solution (3.4) reduces to

$$\Phi = \sum_{f=-\infty}^{\infty} \alpha_f \, \mathrm{e}^{-j2\pi ft}$$

$$= \alpha_0 + \sum_{f=1}^{\infty} (a_f \cos 2\pi ft + b_f \sin 2\pi ft) \qquad \text{(A.12)}$$

which is similar to (A.11) and is indicative of the similarity of the waveforms (Cohen, 1972).

Glossary

E	The electric field
σ	The conductivity of the medium
ϵ	The permittivity of the medium
D	The electrical displacement
ρ	The charge density
C	The capacitivity of the passive elements of the medium
J	The current density
ϕ	The potential